Hand Out

Examples of linear transformation :-

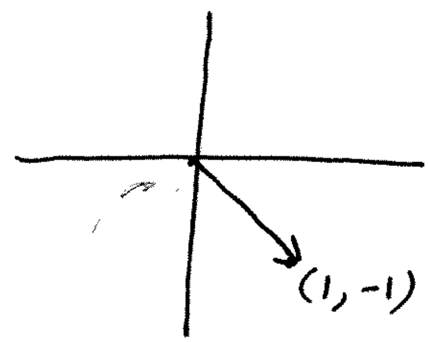
Example 1

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$(x_1, x_2) \longmapsto (x_1 + x_2, 2x_1 + 2x_2, 3x_1 + 3x_2)$$

(a) • Find the null space

$$\left.\begin{array}{l} x_1 + x_2 = 0 \\ 2x_1 + 2x_2 = 0 \\ 3x_1 + 3x_2 = 0 \end{array}\right\} \Rightarrow x_1 = -x_2$$



$$\mathcal{N}(T) = \{(x_1, x_2) : x_1 = -x_2\}$$
$$= \text{span}[(1, -1)]$$

$\mathcal{N}(T)$ is a line with equation $y = -x$.

(b) • Find the range space

Note that

$$\begin{pmatrix} x_1 + x_2 \\ 2x_1 + 2x_2 \\ 3x_1 + 3x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (x_1 + x_2)$$

Hence

$$\mathcal{R}(T) = \{(x, y, z) : y = 2x, z = 3x\}$$

$\mathcal{R}(T)$ is a line in $\mathbb{R}^3$.

$$= \text{span}[(1\ 2\ 3)]$$

- Find the image of a line

(c) $l \triangleq \{3x = 4y\}$

$(4,3)$ is a point in l.

$$T\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}7\\14\\21\end{pmatrix}$$

$$T\left(\alpha\begin{pmatrix}4\\3\end{pmatrix}\right) = \begin{pmatrix}7\alpha\\14\alpha\\21\alpha\end{pmatrix}$$

image of l is $\operatorname{span}\begin{pmatrix}1\\2\\3\end{pmatrix}$ which is the line

$y = 2x,\ z = 3x$ in $\mathbb{R}^3$.

(d) • Find the image of a line

$l \triangleq \{(x,y) : 3x - 4y = 5\}$

l is parameterized as

$$\begin{pmatrix}-5\\-5\end{pmatrix} + t\begin{pmatrix}4\\3\end{pmatrix}$$

$T(l)$ is parameterized as

$$T\begin{pmatrix}-5\\-5\end{pmatrix} + t\,T\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix}(-10) + \begin{pmatrix}1\\2\\3\end{pmatrix}8t = \begin{pmatrix}1\\2\\3\end{pmatrix}(8t-10)$$

It follows that $T(\ell)$ is same as the range of T, ie $R(T)$.

- Calculate the preimage of a point

(e) $p = (3, 6, 9)$
in $\mathbb{R}^3$.

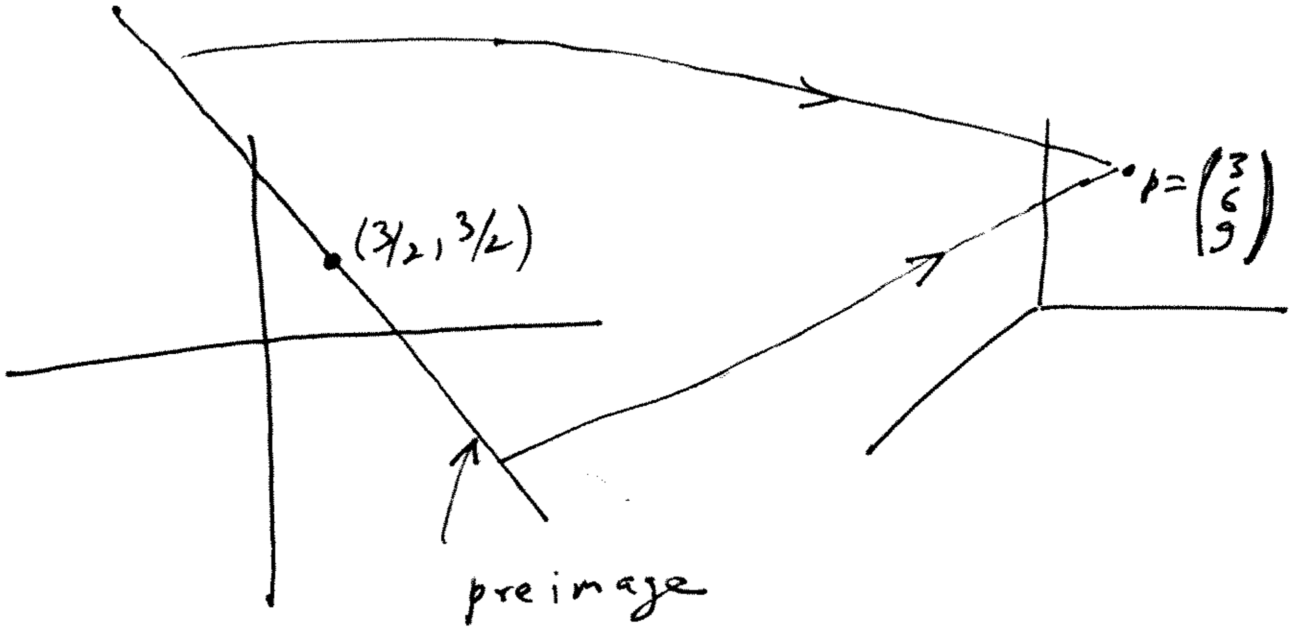
Note: In order to calculate the preimage of a point p, it must be in the range of T. Check that p is indeed in the range of T.

$$T^{-1}(p) \triangleq \left\{ (x_1, x_2) : T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = p \right\}.$$

Preimages are calculated by obtaining one element in the preimage and adding the nullspace to that.

Let $q = \begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix}$. Check that $T(q) = p$.

$$\therefore T^{-1}(p) = \left\{ (x_1 + 3/2, x_2 + 3/2) : x_1 = -x_2 \right\}$$



Note: Preimage of a non-zero point is not a vector space.

★ We shall see in the next section that image and preimage of a vector space is a vector space

Example 2

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 - 3x_2 \end{pmatrix}$$

- Find the null space

$$\begin{aligned} x_1 + x_2 &= 0 \\ x_1 - x_2 &= 0 \\ 2x_1 - 3x_2 &= 0 \end{aligned} \Rightarrow x_1 = x_2 = 0$$

$\boxed{\text{Hence the nullspace is trivial}}$

- Find the range space.

$$R(T) = \{(\alpha+\beta, \alpha-\beta, 2\alpha-3\beta) : \alpha, \beta \in \mathbb{R}\}$$

To calculate a basis of $R(T)$, write down a basis of the domain, viz.

$$\left\{ \underset{e_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \right\}.$$

Now calculate

$$T(e_1) = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad T(e_2) = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$$R(T) = \operatorname{span}\left[\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} \right]$$

To calculate cartesian eqn of $R(T)$ calculate a vector perp. to $R(T)$. Let

$$\left.\begin{aligned} (\theta_1 \ \theta_2 \ \theta_3)\, T(e_1) &= 0 \\ (\theta_1 \ \theta_2 \ \theta_3)\, T(e_2) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} \theta_1 + \theta_2 + 2\theta_3 &= 0 \\ \theta_1 - \theta_2 - 3\theta_3 &= 0 \end{aligned}$$

$$\Downarrow$$

$$\boxed{\theta_2 = \theta_1 - 3\theta_3 = \theta_1 - 6\theta_1 = -5\theta_1} \quad 2\theta_1 - \theta_3 = 0 \ \text{ ie } \boxed{\theta_3 = 2\theta_1}$$

$(1, -5, 2)$ is a vector $\perp$ to both $T(e_1)$ & $T(e_2)$. Hence

$$x - 5y + 2z = 0$$

is a cartesian eqn of the range $R(T)$.

$$R(T) = \{(x, y, z) : x - 5y + 2z = 0\}.$$

(c) Find the image of a line

$$\ell = \{(x, y) : 3x = 4y\}.$$

Calculate $T(\ell)$.

Note that since ℓ is a vector subspace of $\mathbb{R}^2$, $T(\ell)$ is a vector subspace of $\mathbb{R}^3$.

Since $\left\{\begin{pmatrix}4\\3\end{pmatrix}\right\}$ is a basis of ℓ, it follows that

$$T\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}7\\1\\-1\end{pmatrix}$$ is a basis of $T(\ell)$.

$$T(\ell) = \text{span}\left\{\begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix}\right\}$$

$$= \{(x, y, z) : x = 7y, z = -y\}.$$

(d) Find the image of a line

$$\ell \triangleq \{(x, y) : 3x - 4y = 5\}.$$

Recall that ℓ is parameterized as

$$\begin{pmatrix} -5 \\ -5 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

It would follow that $T(\ell)$ is parameterized as

$$T\begin{pmatrix} -5 \\ -5 \end{pmatrix} + t\, T\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -10 \\ 0 \\ 5 \end{pmatrix} + t\begin{pmatrix} 7 \\ 1 \\ -1 \end{pmatrix}$$

This is an affine line in $\mathbb{R}^3$.

To calculate cartesian eqn of $T(l)$, write

$$x = -10 + 7t, \quad y = t, \quad z = 5 - t \implies x = -10 + 7y, \quad z = 5 - y.$$

$$T(l) = \{(x, y, z) : x = -10 + 7y, z = 5 - y\}.$$

(e) Calculate the preimage of a point

$$p = (3, 1, 1).$$

Let (x_1, x_2) be in the preimage, it follows that

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ 2x_1 - 3x_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

Hence $2x_1 = 4 \implies x_1 = 2$.

$\implies x_2 = 3 - x_1 = 1$.

$\therefore \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is in the preimage.

Since Null space of T is trivial, it follows that $(2, 1)$ is the only element in the preimage.

homogeneous.

(f) Let l be a line in $\mathbb{R}^3$ described in cartesian co-ordinates as follows.

$$2x+y+z=0$$
$$3x-4y+3z=0$$

Calculate the preimage of l.

Note that l is a vector subspace of $\mathbb{R}^3$ because it is homogeneous. Let us first calculate a basis of l.

write
$$2x+y+z=0$$
as
$$8x+4y+4z=0$$
$$3x-4y+3z=0$$

$$11x+7z=0 \Rightarrow x=7, z=-11$$

$$y=-2x-z=-14+11=-3$$

$\{(7,-3,-11)\}$ is a basis of l.

$$\ell \triangleq \operatorname{span}\left\{\begin{pmatrix}7\\-3\\-11\end{pmatrix}\right\}.$$

$$T^{-1}(\ell) = \operatorname{span}\left\{\underbrace{T^{-1}\begin{pmatrix}7\\-3\\-11\end{pmatrix}}\right\}$$

Note that this is a pt in $\mathbb{R}^2$

Let $\begin{pmatrix}x\\y\end{pmatrix}$ be such that $T\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}7\\-3\\-11\end{pmatrix}$

we have

$x+y=7 \Rightarrow 2x=4 \Rightarrow x=2.$

$x-y=-3 \qquad y=5$

$2x-3y=-11$

$$\therefore T^{-1}(\ell) = \operatorname{span}\left\{\begin{pmatrix}2\\5\end{pmatrix}\right\}$$

9 Let l be an affine line in $\mathbb{R}^3$ described as follows

$$2x + y + z = 3$$
$$3x - 4y + 3z = 3$$

Calculate the preimage of l.

Note: l is not a vector space. It does not make sense to calculate a basis of l. It does make sense to parametrize the line l, however. Let us do just that.

Find one point on l:

Choose $z = 1 \Rightarrow 2x + y + 1 = 3$
$3x - 4y + 3 = 3$.

$\Downarrow$

$$\begin{aligned} 2x + y &= 2 \\ 3x - 4y &= 0 \end{aligned} \Rightarrow \begin{aligned} 8x + 4y &= 8 \\ 3x - 4y &= 0 \end{aligned}$$

$\Downarrow$

$4y = 3x = \frac{24}{11}$; $\boxed{y = \frac{6}{11}}$ $\quad 11x = 8 \quad \boxed{x = 8/11}$

$\left(\frac{8}{11}, \frac{6}{11}, 1\right)$ is a point in ℓ.

ℓ is thus parametrized as.

$$\begin{pmatrix} \frac{8}{11} \\ \frac{6}{11} \\ 1 \end{pmatrix} + t \begin{pmatrix} 7 \\ -3 \\ -11 \end{pmatrix}$$

Need to calculate this

$T^{-1}(\ell)$ is parametrized as

$$T^{-1}\begin{pmatrix} 8/11 \\ 6/11 \\ 1 \end{pmatrix} + t \underbrace{T^{-1}\begin{pmatrix} 7 \\ -3 \\ -11 \end{pmatrix}}_{= \begin{pmatrix} 2 \\ 5 \end{pmatrix}}$$

Let $\begin{pmatrix} x \\ y \end{pmatrix}$ be such that

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8/11 \\ 6/11 \\ 1 \end{pmatrix}$$

$$\begin{aligned} x+y &= 8/11 \\ x-y &= 6/11 \quad \Rightarrow 2x = \frac{14}{11} \Rightarrow x = 7/11. \\ 2x-3y &= 1 \quad \Rightarrow 2y = 2/11 \Rightarrow y = 1/11 \end{aligned}$$

$T^{-1}(\ell)$ is given by

$$\begin{pmatrix} \frac{7}{11} \\ 1/11 \end{pmatrix} + t\begin{pmatrix} 2 \\ 5 \end{pmatrix} \text{ ie } \begin{pmatrix} \frac{7}{11}+2t \\ \frac{1}{11}+5t \end{pmatrix}$$

Cartesian eqn

$$\begin{aligned} x &= \frac{7}{11}+2t \\ y &= 1/11+5t \end{aligned} \Rightarrow \begin{aligned} 5x &= \frac{35}{11}+10t \\ 2y &= \frac{2}{11}+10t \end{aligned} \Rightarrow 5x-2y = 3.$$

(h) Calculate image of a circle

$$\mathcal{C} = \{x_1^2 + x_2^2 = 16\}.$$

under the transformation T.

Let $S = T(\mathcal{C})$ it follows that

$$S = \{(\alpha, \beta, \gamma) : \alpha = x_1 + x_2, \beta = x_1 - x_2, \gamma = 2x_1 - 3x_2 \ \& \ x_1^2 + x_2^2 = 16\}.$$

Since $(\alpha, \beta, \gamma) \in R(T)$ it follows from (b)

that $\boxed{\alpha - 5\beta + 2\gamma = 0}$

Moreover since

$$x_1 + x_2 = \alpha \Rightarrow 2x_1 = \alpha + \beta \Rightarrow x_1 = \frac{\alpha + \beta}{2}.$$

$$x_1 - x_2 = \beta \qquad 2x_2 = \alpha - \beta \Rightarrow x_2 = \frac{\alpha - \beta}{2}.$$

$$\therefore \left(\frac{\alpha + \beta}{2}\right)^2 + \left(\frac{\alpha - \beta}{2}\right)^2 = 16$$

$$\Rightarrow (\alpha+\beta)^2 + (\alpha-\beta)^2 = 8^2.$$

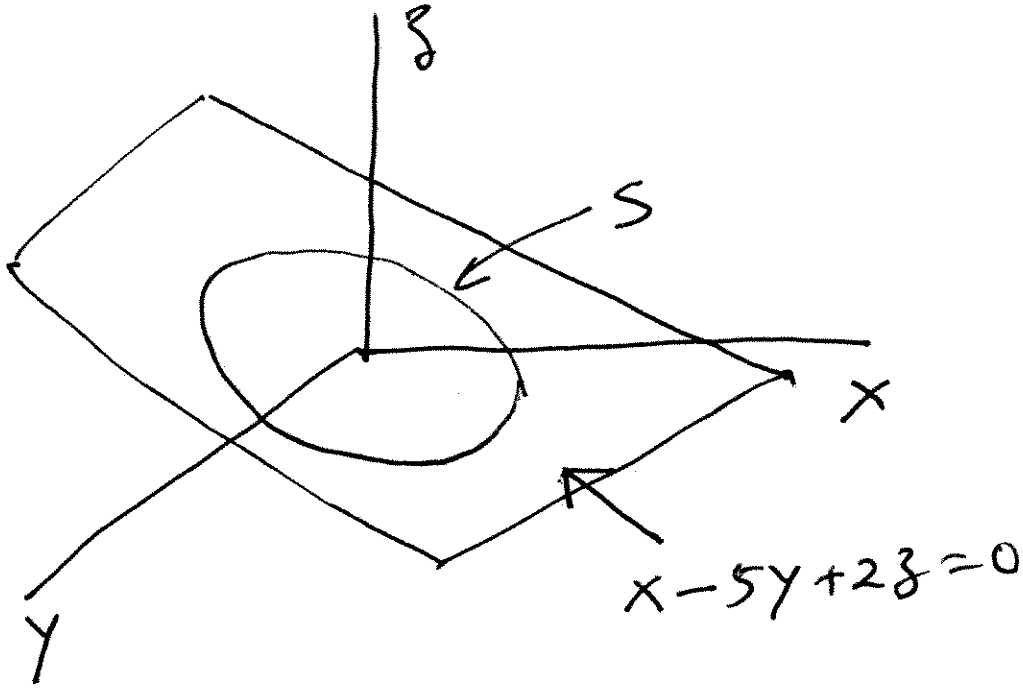
$$\Rightarrow \alpha^2+\beta^2+\alpha^2+\beta^2 = 8^2.$$

$$\Rightarrow \alpha^2+\beta^2 = \left(\frac{8}{\sqrt{2}}\right)^2.$$

Thus S is described in co-ordinates (x,y,z) as

$$x - 5y + 2z = 0$$
$$x^2 + y^2 = (8/\sqrt{2})^2$$

It looks like the projection of S on the (x,y) plane is a circle.



We still don't know how S looks like.

Could it be an ellipse??